

# BIPARTITE CHEBYSHEV POLYNOMIALS AND ELLIPTIC INTEGRALS EXPRESSIBLE BY ELEMENTARY FUNCTIONS

KAZUTO ASAI

*Center for Mathematical Sciences, University of Aizu,*

*Aizu-Wakamatsu, Fukushima 965-8580, Japan*

e-mail: k-asai@u-aizu.ac.jp

*Tel. 0242-37-2644 (Office), 0242-37-2752 (Fax)*

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## Abstract

The article is concerned with polynomials  $g(x)$  whose graphs are “partially packed” between two horizontal tangent lines. We assume that most of the maximum points of  $g(x)$  are on the first horizontal line, and most of the minimum points on the second horizontal line, except several “exceptional” maximum or minimum points, that locate above or under two lines, respectively. In addition, the degree of  $g(x)$  is exactly the number of all extremum points +1. Then we call  $g(x)$  a multipartite Chebyshev polynomial associated with the two lines.

Under a certain condition, we show that  $g(x)$  is expressed as a composition of the Chebyshev polynomial and a polynomial defined by the  $x$ -component data of the exceptional extremum points of  $g(x)$  and the intersection points of  $g(x)$  and the two lines. Especially, we study in detail bipartite Chebyshev polynomials, which has only one exceptional point, and treat a connection between such polynomials and elliptic integrals.

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## 1. INTRODUCTION

The article is concerned with construction of real polynomials  $g(x)$  whose graphs are partially packed in two horizontal tangent lines. Let  $l_1, l_2$  be horizontal lines arranged downwards. We consider the case that most of the maximum points of  $g(x)$  are on  $l_1$  and most of the minimum points on  $l_2$ , except that several maximum or minimum points, which we call the exceptional maximum or minimum points, are located above or under the two lines, respectively. Suppose the degree of  $g(x)$  is exactly the number of all extremum points +1. We call  $g(x)$  a multipartite Chebyshev polynomial associated with the two lines. The simplest case is that  $g(x)$  has no exceptional extremum points, then  $g(x)$  is essentially the Chebyshev polynomial  $T_n(x)$ .

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First of all, we consider the above-mentioned simplest case. Let  $m$  be a positive real constant. Let the horizontal tangent lines  $l_1, l_2$  of  $g(x)$  be  $y = \pm m$  without loss of generality. Suppose  $g(x)$  has no exceptional extremum points.  $g(x) - m$  has zeros at every maximum point of  $g(x)$ , and also  $g(x) + m$  has zeros at every minimum point of  $g(x)$ . Additional two zeros of  $g(x) - m$  or  $g(x) + m$  exist, and we can set them  $x = \pm a$  without loss of generality. Hence we have

$$n^2(g^2 - m^2) = (x^2 - a^2)g'^2, \quad (1)$$

and therefore

$$\int \frac{dg}{\sqrt{m^2 - g^2}} = \pm n \int \frac{dx}{\sqrt{a^2 - x^2}}. \quad (2)$$

Letting  $g = m \cos \varphi$ ,  $x = a \cos \theta$ , noting that  $|x| \rightarrow a$  implies  $|g| \rightarrow m$ , we have

$$-\varphi = \mp n\theta + k\pi. \quad (k \in \mathbb{Z}) \quad (3)$$

$$\therefore g = \pm m \cos \left( n \arccos \frac{x}{a} \right) = \pm m T_n \left( \frac{x}{a} \right).$$

We can treat similarly a multipartite Chebyshev polynomial  $g(x)$  of degree  $n$ . Let the horizontal tangent lines  $l_1, l_2$  of  $g(x)$  be  $y = \pm m$ . Let  $\alpha_1, \dots, \alpha_r$  be the  $x$ -components of the intersection (not tangent) points of the graph of  $g(x)$  and  $l_1$  or  $l_2$ , and let  $\beta_1, \dots, \beta_\ell$  be the  $x$ -components of the exceptional extremum points of  $g(x)$  outside the two lines. By definition,  $r$  is even and  $r = 2\ell + 2$ . For convenience, we call the data  $(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_\ell)$  the outside data of  $g(x)$ , and set  $p(x) = (x - \alpha_1) \dots (x - \alpha_r)$ ,  $q(x) = (x - \beta_1) \dots (x - \beta_\ell)$ . We have

$$n^2(q(x))^2(g^2 - m^2) = p(x)g'^2. \quad (4)$$

Hence as above,

$$\int \frac{dg}{\sqrt{m^2 - g^2}} = \pm n \int \frac{q(x)}{\sqrt{-p(x)}} dx. \quad (5)$$

In general, the RHS is a hyper-elliptic integral. Also, there are no assurance that (4) or (5) has a polynomial solution  $g(x)$ . Now we assume that for some positive divisor  $s$  of  $n$ , there exists a multipartite Chebyshev polynomial  $u(x)$  of degree  $s$  that shares the outside data with  $g(x)$ . Then  $u$  satisfies the same equation as (4):

$$s^2(q(x))^2(u^2 - m^2) = p(x)u'^2. \quad (6)$$

By (4),(6), noting again that  $x \rightarrow \alpha_i$  implies  $|g|, |u| \rightarrow m$ , and letting  $g = m \cos \varphi$ ,  $u = m \cos \theta$ ,

$$\begin{aligned} -\varphi &= \int \frac{dg}{\sqrt{m^2 - g^2}} = \pm \frac{n}{s} \int \frac{du}{\sqrt{m^2 - u^2}} \\ &= \mp \frac{n}{s} \theta + k\pi. \quad (k \in \mathbb{Z}) \\ \therefore g &= \pm m \cos \left( \frac{n}{s} \arccos \frac{u}{m} \right) = \pm m T_{\frac{n}{s}} \left( \frac{u}{m} \right). \end{aligned} \quad (7)$$

Equations (4),(6) are necessary conditions for  $g$  and  $u$ , respectively, and they assure that the non-exceptional extremum points of them are located on  $l_1, l_2$ , but do not assure the total number of such points. In this case, however, as  $u$  in (7) is defined to be a multipartite Chebyshev polynomial, one can confirm that the total number

of extremum points of  $g$  is  $(n/s)s - 1 = n - 1$ , which ensures  $g$  to be a multipartite Chebyshev polynomial as desired.

**Theorem 1.** *Let  $n$  be a positive integer and  $s$  be a positive divisor of  $n$ . Let  $g$  and  $u$  be multipartite Chebyshev polynomials of degree  $n$  and  $s$ , respectively, associated with the lines  $y = \pm m$ , and sharing a common outside data. Then we have  $g = \pm m T_{\frac{n}{s}}(\frac{u}{m})$ .*

## 2. BIPARTITE CHEBYSHEV POLYNOMIALS

A multipartite Chebyshev polynomial  $g(x)$  of degree  $n$  with a unique exceptional extremum point is called a bipartite Chebyshev polynomial. In this case there are four intersection points of  $g(x)$  and two tangent lines  $y = \pm m$ . Let the outside data of  $g$  be  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4; 0)$  ( $\alpha_1 > \alpha_2 > 0 > \alpha_3 > \alpha_4$ ) without loss of generality. If the data satisfies symmetric condition:  $\alpha_1 = -\alpha_4$ ,  $\alpha_2 = -\alpha_3$ , and  $n$  is even, then  $g$  is a special case of (7) with a quadratic  $u$ , say,  $g = \pm m T_{\frac{n}{2}}\left(\frac{2x^2 - \alpha_1^2 - \alpha_2^2}{\alpha_1^2 - \alpha_2^2}\right)$ .

We now proceed to the case without symmetric condition. In accordance with the outside data, we have

$$n^2 x^2 (g^2 - m^2) = p(x) g'^2. \quad (8)$$

Let  $s$  be a positive divisor  $s$  of  $n$  (possibly  $s = n$ ) such that there exists a bipartite Chebyshev polynomial  $u$  of degree  $s$  sharing a common outside data with  $g$ , then

$$s^2 x^2 (u^2 - m^2) = p(x) u'^2. \quad (9)$$

By Theorem 1,  $g$  is represented as  $g = \pm m T_{\frac{n}{s}}(\frac{u}{m})$ . Thus we study the solution  $u$  to equation (9). Since  $u'(0) = 0$ , we can set

$$u = a_0 + a_2 x^2 + a_3 x^3 + \cdots + a_s x^s. \quad (10)$$

Dividing both sides of (9) by  $x^2$  and differentiating with respect to  $x$ , we have

$$2s^2 u = 2u'' \tilde{p} + u' \tilde{p}', \quad (11)$$

where  $\tilde{p} = p(x)/x^2$ . It follows from (11), by setting  $\tilde{p} = x^2 + c_1 x + c_2 + c_3 x^{-1} + c_4 x^{-2}$  and comparing the coefficients of  $x^k$ , that

$$a_k = \frac{1}{2(s^2 - k^2)} \sum_{i=1}^4 (k+i)(2k+i) c_i a_{k+i} \quad (k = 0, 1, \dots, s-1), \quad (12)$$

where we promise  $a_1 = a_{s+1} = a_{s+2} = a_{s+3} = 0$ . By (12), the coefficients are determined step by step from  $a_s$  to  $a_3, a_2, a_0$  in the form:

$$a_k = F_k(c_1, c_2, c_3, c_4) a_s, \quad (13)$$

where  $F_k$  is a polynomial in  $c_1, c_2, c_3, c_4$  with rational coefficients for every  $k = 0, 2, 3, \dots, s$ . The rest of the condition (9) is the equation (13) for  $k = 1$ :

$$F_1(c_1, c_2, c_3, c_4) = 0, \quad (14)$$

and the  $x^2$  terms of (9):

$$s^2 (a_0^2 - m^2) = 4a_2^2 c_4 \iff (s^2 F_0^2 - 4c_4 F_2^2) a_s^2 = s^2 m^2. \quad (15)$$

**Lemma 1.** *For every  $k = 0, 1, \dots, s$ , the polynomial  $F_k = F_k(c_1, c_2, c_3, c_4)$  has the following properties:*

- (i) *The coefficient of every term of  $F_k$  is positive.*
- (ii)  *$F_{s-k}$  consists of the terms of type  $c_{i_1}c_{i_2}\dots c_{i_r}$ , such that  $i_1 + i_2 + \dots + i_r = k$ .*
- (iii) *Every possible term of type  $c_{i_1}c_{i_2}\dots c_{i_r}$  with  $i_1 + i_2 + \dots + i_r = k$  appears in  $F_{s-k}$  except for  $F_0$  does not contain the term  $c_1^s$ .*

*Proof.* For convenience, write  $c_{i_1}c_{i_2}\dots c_{i_r} = c_\lambda$ , where  $\lambda = (\lambda_1, \dots, \lambda_r)$  is the weakly decreasing rearrangement of  $i_1, \dots, i_r$ . Then  $\lambda$  is a partition of  $k$  (denoted by  $\lambda \vdash k$ ),  $k = i_1 + \dots + i_r$ . Each  $\lambda_i$  is called a part of  $\lambda$ , and  $r$  is called the length of  $\lambda$  denoted by  $\ell(\lambda)$ . Noting that all coefficients of the linear recurrence relation (12) are positive, we see (i). Next, we prove (ii) and (iii) by induction on  $k$ . When  $k = 0$ , we have  $F_s = 1$  which satisfies the proposition. For the induction step, suppose, for every nonnegative integer  $j < k$ ,

$$F_{s-j} = \sum_{\lambda \vdash j; \lambda_1 \leq 4} m_\lambda c_\lambda \quad (16)$$

with some positive coefficients  $m_\lambda$ . Then by (12), putting  $\frac{(s-k+i)(2s-2k+i)}{2k(2s-k)} = m_i^{(k)}$ ,

$$\begin{aligned} F_{s-k} &= \sum_{i=1}^4 m_i^{(k)} c_i F_{s-k+i} = \sum_{i=1}^4 \sum_{\lambda \vdash k-i; \lambda_1 \leq 4} m_i^{(k)} m_\lambda c_i c_\lambda \\ &= \sum_{\lambda \vdash k; \lambda_1 \leq 4} m_\lambda c_\lambda. \end{aligned} \quad (17)$$

Here, as every partition  $\lambda$  of  $k$  with  $\lambda_1 \leq 4$  is composed of the parts  $\lambda_j$  such that  $1 \leq \lambda_j \leq 4$ , we can reduce the partition  $\lambda$  to a partition of  $k-i$  by subtracting some part  $1 \leq i \leq 4$ . Therefore, together with the positivity of the coefficients  $m_i^{(k)} m_\lambda$ , every coefficient  $m_\lambda$  is shown to be positive. Only the case  $F_0$ , however, is represented as

$$F_0 = \sum_{i=2}^4 m_i^{(s)} c_i F_i = \sum_{i=2}^4 \sum_{\lambda \vdash s-i; \lambda_1 \leq 4} m_i^{(s)} m_\lambda c_i c_\lambda, \quad (18)$$

because  $a_1$  is defined to be 0. From this, it follows that  $F_0$  does not contain only the term  $c_{(1,\dots,1)} = c_1^s$ .  $\square$

Reviewing (17), we have  $m_\lambda = \sum_{i=1}^4 m_i^{(k)} m_{\lambda \ominus (i)}$  for  $\lambda \vdash k$ , where  $\lambda \ominus (i)$  denotes a partition obtained by removing a part of size  $i$  from  $\lambda$ , and set  $m_{\lambda \ominus (i)} = 0$  in case that  $\lambda$  has no parts of size  $i$ . Iterating this induction, we can represent the coefficient  $m_\lambda$  as follows.

**Lemma 2.** *Let  $S(\lambda)$  denote the set of all distinct permutations  $i = (i_1, \dots, i_r)$  of  $\lambda$ . If  $\lambda$  is a partition of the integer less than  $s$ , then*

$$m_\lambda = \sum_{i \in S(\lambda)} \prod_{t=1}^r m_{i_t}^{(i_1 + \dots + i_t)}, \quad (19)$$

while if  $\lambda \vdash s$ ,  $S(\lambda)$  in this formula should be replaced with  $S'(\lambda)$ , the set of all distinct permutations  $(i_1, \dots, i_r)$  of  $\lambda$  where  $i_r > 1$ .

On the basis of the above-mentioned method, we construct the polynomial  $u$  as described in the procedure below:

- (i) Choose an arbitrary positive constant  $m$ , and arbitrary real coefficients  $c_2, c_3, c_4$  of  $p$  such that  $c_2 < 0 < c_4$ .
- (ii) Represent all coefficients  $a_k$  of  $u$  as a polynomial in  $c_1, \dots, c_4$  and  $a_s$  by using (13), (16) and (19).
- (iii) By (14), determine  $c_1$  depending on  $c_2, c_3, c_4$ .
- (iv) Determine  $a_k$  for all  $k = 0, 2, 3, \dots, s-1$ .
- (v) Also determine  $a_s$  by (15).

Unfortunately, in the steps (iii) and (v),  $c_1$  is possibly an imaginary number, and  $a_s$  may be also imaginary, or even does not exist if  $s^2 F_0^2 - 4c_4 F_2^2 = 0$ . Now we avoid these difficulties by handling the coefficients  $c_3, c_4$  ( $c_2 < 0$  is fixed). Letting  $c_3, c_4 = 0$ , equation (9) becomes

$$s^2 x^2 (u^2 - m^2) = x^2 (x^2 + c_1 x + c_2) u'^2, \quad (20)$$

which is clearly reduced to (1), and we have  $u = \pm m T_s \left( \frac{x-b}{a} \right)$ . But by the continuity of all equations in the above process (i)–(v), we can show that for  $c_3$  and  $c_4$  sufficiently close to 0,  $c_1$  and  $a_s$  are determined to be real numbers. Equation (20) has  $s-1$  possible values of  $c_1$  for given  $c_2$ , because the condition (14), i.e.,  $u'(0) = 0$  obliges one of the extremum points of  $u$  to be located on the  $y$ -axis, which allows  $s-1$  possible graphs of  $u = u_1, \dots, u_{s-1}$ . While by Lemma 1,  $F_1$  has a leading term  $c_1^{s-1}$  with respect to  $c_1$ , and this has  $s-1$  distinct real roots  $c_1$  for  $c_3 = c_4 = 0$ . Hence, again for  $c_3$  and  $c_4$  sufficiently close to 0, by the continuity of  $F_1$ , we have  $s-1$  distinct real values of  $c_1$ , and therefore  $s-1$  different  $u$ 's (also possible two  $\pm a_s$ 's), that are obtained graphically from  $u_1, \dots, u_{s-1}$  moving slightly the maximum or minimum point on  $y$ -axis upward or downward, respectively. Reducing the expression  $g = \pm m T_{\frac{n}{s}} \left( \frac{u}{m} \right)$  to  $\pm m T_{\frac{n}{s}}(u)$ , we have the following.

**Theorem 2.** *For a positive integer  $n$  and a positive divisor  $s$  of  $n$ , there exists a bipartite Chebyshev polynomial of degree  $n$  of the form  $g = \pm m T_{\frac{n}{s}}(u)$  associated with the lines  $y = \pm m$ , where  $u$  is also a bipartite Chebyshev polynomial of degree  $s$  associated with the lines  $y = \pm 1$  obtained by moving an arbitrary one of the maximum or minimum points of  $\pm T_s \left( \frac{x-b}{a} \right)$  upward or downward, respectively.*

### 3. ELLIPTIC INTEGRALS

Focusing again on the integral solution of (8), a special case of Solution (5), for a monic quartic polynomial  $p(x)$ ,

$$\begin{aligned} \int \frac{x}{\sqrt{-p(x)}} dx &= \pm \frac{1}{n} \arccos \frac{g}{m} + C \quad (p(x) < 0) \\ \int \frac{x}{\sqrt{p(x)}} dx &= \pm \frac{1}{n} \operatorname{arccosh} \frac{g}{m} + C \quad (p(x) > 0). \end{aligned} \quad (21)$$

Hence, if  $g$  is a polynomial (which should be of degree  $n$ ), then the elliptic integral on the LHS is represented as an elementary function. We already know that the condition

for  $g$  to be a polynomial is (14) and (15) for some divisor  $s$  of  $n$ . By Section 2, for every positive integer  $n$ , there exists a polynomial  $g$  of degree  $n$  under suitable condition for  $p(x)$ , and in this case we obtain a bipartite Chebyshev polynomial  $g$ . However, the assumption for  $p(x)$  to represent the outside data is not necessary for an argument for the representation of the integration, we just need (14), (15) for this purpose.

Forgetting the connection with multipartite Chebyshev polynomials, we can also deal with the equation similar to (8) for a polynomial  $g$  of degree  $n$ :

$$n^2 x^2 (g^2 + m^2) = p(x) g'^2. \quad (22)$$

In the same way as in Section 2, for some positive divisor  $s$  of  $n$ , suppose there exists a polynomial  $u$  of degree  $s$  such that

$$s^2 x^2 (u^2 + m^2) = p(x) u'^2. \quad (23)$$

Letting  $g = m \sinh \varphi$ ,  $u = m \sinh \theta$  temporarily in the complex domain (to determine the constant of integration), and noting that  $p(x) \rightarrow 0$  implies  $g, u \rightarrow \pm mi$ , we have

$$\begin{aligned} \varphi &= \int \frac{dg}{\sqrt{m^2 + g^2}} = \pm \frac{n}{s} \int \frac{du}{\sqrt{m^2 + u^2}} = \pm \frac{n}{s} \theta. \\ \therefore g &= \pm m \sinh \left( \frac{n}{s} \operatorname{arcsinh} \frac{u}{m} \right) \equiv \pm m \tilde{T}_{\frac{n}{s}} \left( \frac{u}{m} \right). \end{aligned} \quad (24)$$

Here, if  $g, u \rightarrow \pm mi$ , then  $\varphi \rightarrow (\frac{\pi}{2} + k\pi)i$ ,  $\theta \rightarrow (\frac{\pi}{2} + k'\pi)i$  ( $k, k' \in \mathbb{Z}$ ), but this implies that  $n/s$  is odd, which we now assume, and the (real) constant of integration vanishes. One sees that  $\tilde{T}_n(x) = \sinh(n \operatorname{arcsinh} x)$  is monotonously increasing, and a polynomial of degree  $n$  iff  $n$  is an odd positive integer.

Returning to (23), a change from (9) to (23) does not have an effect on the condition (14), while it turns (15) into

$$s^2(a_0^2 + m^2) = 4a_2^2 c_4 \iff (4c_4 F_2^2 - s^2 F_0^2) a_s^2 = s^2 m^2. \quad (25)$$

For convenience, set  $d = s^2 F_0^2 - 4c_4 F_2^2$ . We see the signature of  $d$  discriminates the existence of  $a_s$  in (15) or (25). On the other hand, the solution of (22) is represented as

$$\int \frac{x}{\sqrt{p(x)}} dx = \pm \frac{1}{n} \operatorname{arcsinh} \frac{g}{m} + C. \quad (26)$$

For the special case  $m = 0$ , one sees that

$$\int \frac{x}{\sqrt{p(x)}} dx = \pm \frac{1}{n} \log |g| + C; \quad g = cu^{n/s}. \quad (27)$$

**Theorem 3.** *Let  $n$  be a positive integer. For a quartic polynomial  $p(x) = x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4$ , the elliptic integral*

$$\int \frac{x}{\sqrt{\pm p(x)}} dx \quad (28)$$

*is represented as either (21) or (26) or (27) subject to  $d > 0$  or  $d < 0$  or  $d = 0$ , where  $g$  is a polynomial of degree  $n$  expressed as (7) or (24) or (27), respectively, if and only if (14) is satisfied for some positive divisor  $s$  of  $n$ , such that  $n/s$  is an odd number when  $d < 0$ . ( $u$  is determined by (10), (13), (16), (19) and (15) or (25).)*

*Proof.* We have seen in Sections 2–3 that (14) for some positive divisor  $s$  of  $n$  ( $n/s$  is odd when  $d < 0$ ) implies the representability of the integral as in the theorem. Conversely, suppose the integral is represented as in the theorem, then (8) or (22) or the case  $m = 0$  is satisfied and therefore for  $s = n$ , (14) should be satisfied.  $\square$

Lastly, we study the set of quartic polynomials  $p(x)$  such that (28) is representable as in Theorem 3, depending on the degree  $n$  of  $g$ . For even  $n$ , the polynomial  $F_1(c_1, c_2, c_3, c_4)$  for  $s = n$  has the leading term  $m_{(1, \dots, 1)} c_1^{n-1}$  with respect to  $c_1$ , and thus (as  $n - 1$  is odd), for arbitrary  $c_2, c_3, c_4$ , we can find the real solution  $c_1$  to (14). Therefore for given  $c_2, c_3, c_4$ , we have  $p(x)$  suitable for our representation as in Theorem 3. For  $s = n = 4k - 1$ ,  $F_1$  has the leading term  $m_{(2, \dots, 2)} c_2^{2k-1}$  with respect to  $c_2$ , and therefore for arbitrary  $c_1, c_3, c_4$ , we have  $c_2$  and  $p(x)$  as desired. For  $s = n = 8k - 3$ ,  $F_1$  has the leading term  $m_{(4, \dots, 4)} c_4^{2k-1}$  with respect to  $c_4$ , and so for arbitrary  $c_1, c_2, c_3$ , we have  $c_4$  and  $p(x)$  similarly. The rest case  $n = 8k + 1$  gives no clear assurance for  $p(x)$ , but if  $n$  has some positive divisor not congruent to 1 modulo 8, we have  $p(x)$  suitable for our representation.

**Theorem 4.** *If some positive divisor of  $n$  is not congruent to 1 modulo 8, then for some  $i_1, i_2, i_3$  ( $1 \leq i_1 < i_2 < i_3 \leq 4$ ), given  $c_{i_1}, c_{i_2}, c_{i_3}$ , we obtain  $p(x)$  suitable for representation of (28) as in Theorem 3 by a polynomial  $g$  of degree  $n$ .*

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CENTER FOR MATHEMATICAL SCIENCES, UNIVERSITY OF AIZU, AIZU-WAKAMATSU, FUKUSHIMA 965-8580, JAPAN

*E-mail address:* k-asai@u-aizu.ac.jp